Blackboard

left board:

(1)

Aim: Show that there are Hilbert spaces

(i) $V \overset{d}{\hookrightarrow} H$ and a (ii) bounded, coercive form $a : V \times V \to \mathbb{C}$ such that

nachträglich

(iii) $D(A^{1/2}) \neq V$ where A is the assoc. operator of a.

Remark: The structure of a will look like
$$a(u,v) = \sum_{N \geq 1} a_N(u_N, v_N)$$
with suitable forms a_N.

(7)

Definition:
$$H := \{u = (u_1, u_2, \dots) \in \prod_{N \geq 1} \mathbb{C}^{N+1} \mid \sum_{N \geq 1} \|u_N\|^2_{\mathbb{C}^{N+1}} < \infty\},$$
$$V := \{u \in H \mid \sum_{N \geq 1} \|D_N u_N\|^2 < \infty\},$$
and
$$\langle u, v \rangle_{\underset{V}{H}} := \sum_{N \geq 1} \langle D_N u_N, D_N v_N \rangle_{\mathbb{C}^{N+1}}$$
For $N \in \mathbb{N}$ choose z_N like in (4), set $a_N := a_{N, z_N}$ and a like in the remark.

(2)

Definition: Let $N \in \mathbb{N}$ and $D_N := \operatorname{diag}(2^j)$ and

$B_N := (b_{jk})$ and $C_N := (c_{jk})$ given by
$$b_{jk} := \begin{cases} \frac{i}{\pi(k-j)} & , j \neq k \\ 0 & , j = k \end{cases}, \quad c_{jk} := \begin{cases} \frac{\lambda_j i}{(\lambda_k + \lambda_j)\pi(k-j)} & , j \neq k \\ 0 & , j = k \end{cases}$$
for $0 \leq j, k \leq N$.

$$\frac{2^j}{2^j + 2^k} = \frac{\lambda_j}{\lambda_j + \lambda_k}$$

Vorgehensweise:

- (1) + (2) + (5) verdeckt vorbereitet
- Mittelteile der Tafeln und bei rechter auch am Anfang rechten Außenteil für Beweise nutzen

right board:

simple rule

Claim: (1) D_N, B_N are self-adjoint

streichen

(and $D_N C_N + C_N D_N = D_N B_N$.)

(3)

(2) $\|B_N\| \leq 1$ and
$$\|C_N \tilde{u}_N\|_{\mathbb{C}^{N+1}} \geq \frac{1}{3\pi}(\ln(N) - 2)\|\tilde{u}_N\|$$
with $\tilde{u}_N = (1, \dots, 1)$.

(4)

Definition: For $|z| < 1$ let
$$a_{N,z} : \mathbb{C}^{N+1} \times \mathbb{C}^{N+1} \to \mathbb{C}, \quad a_{N,z}(u,v) := \langle (I + zB_N) D_N u, D_N v \rangle.$$

Claim: (3) $a_{N,z}$ is coercive, in particular
$$\operatorname{Re} a_{N,z}(u) \geq (1 - |z|)\|D_N u\|^2 \quad \forall u,$$
thus the assoc. op. $A_{N,z} = D_N(I + zB_N)D_N$ is m-accretive and $A_{N,z}^{1/2}$ exists.

(5) nur diesen Teil

(4) Let $w_N := D_N^{-1}\tilde{u}_N$. Then:
$$\exists |z_N| = \tfrac{1}{2} : \|A_{N,z_N}^{1/2} w_N\| \geq \tfrac{1}{2} M_N \|\tilde{u}_N\|$$

(6) $\rightsquigarrow M_N := \frac{1}{3\pi}(\ln(N) - 2)$ in (3) definieren

Kein Beweis von (4).

(1) nicht beweisen, mündlich erklären

(2) $\|C_N \tilde{u}_N\| \geq \dots$ mit „geeignet wischen" im Beweis

Remark: Omit index N in some proofs.

Talk „Counter-example" ①

proof: (2) (i) Let $W := \operatorname{span}\{e^{ikt} \mid 0 \le k \le N\} \subset L^2([0,2\pi])$ and $b(t) := -\frac{1}{\pi}t + 1$,

$\mathcal{B} : W \to W,\ \mathcal{B}(f) := P_W(bf)$, P_W orth. proj.

$\Rightarrow \|B\| = \|\mathcal{B}\|_{L(W)} \le \underbrace{\|P\|}_{\le 1} \underbrace{\|b\|_\infty}_{=1} \le 1$

(ii)

$$\pi \|\tilde u_N\| \|C_N \tilde u_N\| \underset{CS}{\ge} \pi |\langle C_N \hat u_N, \hat u_N\rangle| = \left| \sum_{j=0}^{N} \sum_{\substack{k=0\\k\ne j}}^{N} \frac{2^j}{(2^k+2^j)(k-j)} \right|$$

durch ergänzen oben

$$\underset{\text{split into two parts}}{=} \left| \sum_{j=0}^{N} \sum_{k=0}^{j-1} \frac{2^j}{(2^k+2^j)(k-j)} + \sum_{j=0}^{N} \sum_{k=j+1}^{N} \frac{2^j}{(2^k+2^j)(k-j)} \right|$$

$$\underset{\Delta\text{-ineq. from below}}{\ge} \left| \ldots \right| - \left| \sum_{j=0}^{N} \sum \ldots \right|$$

wegwischen, da negativ, und k-j zu j-k ändern

wegwischen, da positiv

$$\underset{\text{factor out } 2^j}{=} \sum_{j=0}^{N} \sum_{k=0}^{j-1} \frac{1}{1+2^{k-j}} \cdot \frac{1}{j-k} - \sum_{j=0}^{N} \sum_{k=j+1}^{N} \frac{1}{1+2^{k-j}} \frac{1}{k-j}$$

wischen

$$\underset{\text{Index ändern}}{=} \text{"} \sum_{k=1}^{j} \underbrace{\frac{1}{1+2^{-k}}}_{\ge \frac{2}{3}} \frac{1}{k} - \sum \sum_{k=1}^{N-j} \underbrace{\frac{1}{1+2^{k}}}_{\le \frac{1}{3}} \frac{1}{k}$$

$\ge \frac{2}{3} \leftarrow$ mit wischen ändern $\to \le \frac{1}{3}$

$$\ge \frac{2}{3} \sum_{j=1}^{N} \overbrace{\sum_{k=1}^{j} \frac{1}{k}}^{\ge \ln(j) + \frac{1}{j}} - \frac{1}{3} \sum_{j=0}^{N-1} \underbrace{\sum_{k=1}^{N-j} \frac{1}{k}}_{\le \ln(N-j)+1}$$

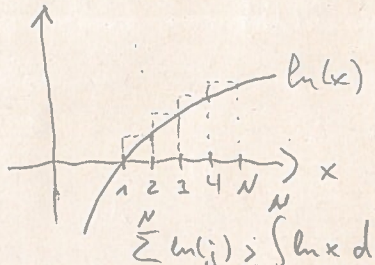
Indizes durch wischen ändern

um auf Indizes verweisen, $\frac{1}{3}$ fällt weg

$$\underset{\text{wischen}}{\ge} \frac{1}{3} \underbrace{\sum_{j=1}^{N} \ln(j)}_{\ge N\ln(N) - N + 1} + \frac{2}{3} \sum_{j=1}^{N} \frac{1}{j} - \frac{1}{3} N$$

dieselbe Absch.

$$\ge \frac{1}{3} (\ln(N) - 2) \underbrace{(N+1)}_{= \|\tilde u_N\|^2}$$



$\sum_{j=1}^{N} \ln(j) \ge \int_1^N \ln x \, dx = x\ln x - x \Big|_1^N$

(3) Let $u \in \mathbb{C}^{N+1}$.

$$\operatorname{Re} a_{N,z}(u) = \operatorname{Re}(\langle Du, Du\rangle + z \underbrace{\langle BDu, Du\rangle}_{\in \mathbb{R},\ B \text{ selfadjoint}})$$

$$\overset{(1)}{=} \|Du\|^2 + \operatorname{Re}(z) \langle BDu, Du\rangle$$

$$\underset{\text{ergänzen wischen}}{\ge} \|Du\|^2 - |z| |\langle BDu, Du\rangle| \overset{CS}{\ge} \|Du\|^2 - |z| \overset{(2)}{\underset{\le 1}{\|B\|}} \|Du\|^2$$

bounded:

$|a_{N,z}(u,v)| \le (1 + |z| \|B\|) \|D\|^2 \|u\| \|v\|$

(4) ②

$$A_z^{1/2} A_z^{1/2} = A_z = D(I+zB)D,$$

$\Longrightarrow$ differentiate both sides, set $z=0$ and use $A_{N,0}^{1/2} = D_N$

$$\underset{\underset{\text{wischen } D!}{\uparrow}}{A_0^{1/2}} \left(\frac{d}{dz} A_z \Big|_{z=0}\right) + \left(\frac{d}{dz} A_z \Big|_{z=0}\right) \underset{\underset{D}{\uparrow}}{A_0^{1/2}} = DBD$$

$$\overset{(1)}{\Longrightarrow} \frac{d}{dz} A_z^{1/2} \Big|_{z=0} = CD$$

$$\underset{\underset{\text{durch Ergänzen}}{\uparrow}}{\Longrightarrow} \left\| \frac{d}{dz} A_{N,z}^{1/2}\Big|_{z=0} \omega_N \right\| = \| C_N D_N \omega_N \| = \| C_N \tilde{u}_N \| \overset{(2)}{\geq} \mu_N \|\tilde{u}_N\|$$

<u>Assume</u>: $\neg(4)$

$$\Longrightarrow \left\| \left(\frac{d}{dz} A_{N,z}^{1/2}\Big|_{z=0}\right) \omega_N \right\| \underset{\text{Cauchy}}{=} \frac{1}{2\pi} \left\| \int_{|z|=\frac{1}{2}} \frac{1}{z^2} A_{N,z}^{1/2} \omega_N \, dz \right\| \leq \frac{1}{2\pi} \underset{\underset{\text{length}}{\uparrow}}{\pi} \left(\frac{1}{4}\right)^{-1} \sup_{|z|=\frac{1}{2}} \| A_{N,z}^{1/2} \omega_N \|$$

$$\underset{\text{assump.}}{<} 2 \cdot \frac{1}{2} \mu_N \|\tilde{u}_N\| \quad \lightning$$

<u>Now back to our aim</u>: Nummern in aim setzen (i), (ii), (iii)

(i) $u \in H$, $v_n := (u_1, \dots, u_n, 0, \dots) \in V$

(ii) $\operatorname{Re} a(u) = \sum_{N \geq 1} \operatorname{Re} a_N(u_N) \underset{(3)}{\geq} \sum_{N\geq 1} (1 - \underbrace{|z_N|}_{=1/2}) \|D_N u\|^2 = \frac{1}{2} \|u\|_j^2$, bounded since $|a(u,v)| \leq \frac{3}{2} \|u\|_V \|v\|_V$

(iii) <u>Assume</u>: $D(A^{1/2}) = V$ topolog.

Let $x_N := (0, \dots, 0, \underset{\underset{N\text{-th comp.}}{\uparrow}}{\frac{\omega_N}{\|\tilde{u}_N\|}}, 0, \dots)$. $\Rightarrow x_N \in V$, $\|x_N\|_V = 1 \ \forall N \in \mathbb{N}$

$$\underset{\text{assump.}}{\Longrightarrow} \|x_N\|_{D(A^{1/2})} = \underbrace{\|x_N\|_H}_{\geq 0 \text{ wegwischen}} + \|A^{1/2} x_N\|_H \geq \left\| A_{N,z_N}^{1/2} \frac{\omega_N}{\|\tilde{u}_N\|} \right\| = \frac{1}{\|\tilde{u}_N\|} \| A_{N,z_N}^{1/2} \omega_N \|$$

$\geq$ ändern

herausziehen wischen.

$$\overset{(4)}{\geq} \frac{1}{2} \mu_N \overset{(2)}{=} \frac{1}{6\pi} (\ln(N) - 2) \longrightarrow \infty, \ N \to \infty \quad \lightning$$