

The death of bi-continuity

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Motivation

Observation

The semigroups $(S(t))_{t \geq 0}$ given by



$$S(t)f(x) := f(x+t), \quad x \in \mathbb{R}, f \in C_b(\mathbb{R}), t \geq 0,$$



$$S(t)f(z) := f(e^{it}z), \quad z \in \mathbb{D}, f \in H^\infty(\mathbb{D}), t \geq 0,$$

▶ $S(0)f := f$ and

$$S(t)f(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{\frac{-|y-x|^2}{4t}} dy, \quad x \in \mathbb{R}^d, f \in C_b(\mathbb{R}^d), t > 0,$$

are **not** strongly continuous w.r.t. $\|\cdot\|_\infty$.

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and the cosine family $(S(t))_{t \geq 0}$ given by



$$S(t)f(x) := \frac{1}{2}(f(x+t) + f(x-t)), \quad x \in \mathbb{R}, f \in C_b(\mathbb{R}), t \geq 0,$$

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are **not** strongly continuous w.r.t. $\|\cdot\|_\infty$.

However, they are strongly continuous w.r.t. the compact-open topology τ_{co} . They belong to the class of **locally bi-continuous** semigroups/cosine families.

Saks space & Bi-continuity

Definition (Wiweger 1961, Cooper 1978)

The triple $(X, \|\cdot\|, \tau)$ is called a **Saks space** if

- ▶ $(X, \|\cdot\|)$ is normed, τ a coarser Hausdorff l.c. topology on X , and
- ▶ there is a system of seminorms Γ_τ inducing τ such that

$$\|x\| = \sup_{p \in \Gamma_\tau} p(x), \quad x \in X.$$

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Definition (Kühnemund '01, Duan & Sun '06, Budde '24)

Let $(X, \|\cdot\|, \tau)$ be a Saks space. A semigroup or cosine family $(S(t))_{t \geq 0}$ in $\mathcal{L}(X)$ is called **locally bi-continuous** if it is

1. τ -strongly continuous,
2. **exponentially bounded**, i.e. $\exists M \geq 1, \omega \in \mathbb{R} \forall t \geq 0 : \|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$,
3. **locally bi-equicontinuous**, i.e.

$$\forall t_0 \geq 0, p \in \Gamma_\tau, (x_n)_{n \in \mathbb{N}} \subset X, x_n \rightarrow 0 \text{ w.r.t. } \tau, \sup_{n \in \mathbb{N}} \|x_n\| < \infty :$$

$$\sup_{t \in [0, t_0]} p(S(t)x_n) \rightarrow 0.$$

Strong continuity and (sequential) equicontinuity

Let

- ▶ X be a locally convex Hausdorff space, and
- ▶ Γ a system of seminorms inducing its topology.

Definition

A family $(S(t))_{t \geq 0}$ of linear maps from X to X is called

- ▶ **semigroup** if $S(0) = \text{id}$ & $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$,
- ▶ **cosine family** if $S(0) = \text{id}$ & $2S(t)S(s) = S(t+s) + S(t-s)$ for all $t \geq s \geq 0$,
- ▶ **strongly continuous** if $[0, \infty) \ni t \mapsto S(t)x \in X$ is continuous for all $x \in X$,

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- ▶ **strongly continuous** if $[0, \infty) \ni t \mapsto S(t)x \in X$ is continuous for all $x \in X$,
- ▶ **locally equicontinuous** if

$$\forall t_0 \geq 0, p \in \Gamma \exists q \in \Gamma, C \geq 0 \forall x \in X : \sup_{t \in [0, t_0]} p(S(t)x) \leq Cq(x),$$

- ▶ **exponentially equicontinuous** if

$$\exists \alpha \in \mathbb{R} \forall p \in \Gamma \exists q \in \Gamma, C \geq 0 \forall x \in X : \sup_{t \geq 0} p(e^{-\alpha t} S(t)x) \leq Cq(x),$$

- ▶ **sequentially locally equicontinuous** if

$$\forall t_0 \geq 0, p \in \Gamma, (x_n)_{n \in \mathbb{N}} \subset X, x_n \rightarrow 0 : \sup_{t \in [0, t_0]} p(S(t)x_n) \rightarrow 0.$$

Mixed topology

Back to the definition of local bi-equicontinuity:

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Observation (Wiweger 1961):

► There is a l.c. Hausdorff topology $\gamma := \gamma(\|\cdot\|, \tau)$ such that:

$$\gamma\text{-}\lim_{n \rightarrow \infty} x_n = x \quad \Leftrightarrow \quad \tau\text{-}\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|x_n\| < \infty$$

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- ▶ γ is the finest linear topology s.t. $\gamma = \tau$ on $\|\cdot\|$ -bounded sets.
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Example (Buck 1958, Cooper 1971)

- ▶ Let $X := C_b(\mathbb{R}^d)$ with $\|\cdot\| := \|\cdot\|_\infty$ and $\tau := \tau_{co}$.
- ▶ Then γ is generated by the seminorms

$$|f|_\nu := \sup_{x \in \mathbb{R}^d} |f(x)\nu(x)|, \quad f \in X, \quad \nu \in C_0(\mathbb{R}^d).$$

Bi-continuity & Mixed topology

Corollary

Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(S(t))_{t \geq 0}$ a semigroup or cosine family on X . Then the following assertions are equivalent.

1. $(S(t))_{t \geq 0}$ is locally bi-continuous.
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Theorem (Kraaij '16)

Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(S(t))_{t \geq 0}$ a semigroup on X .

If (X, γ) is **C-sequential**, then the following assertions are equivalent.

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Definition (Snipes 1973)

Let X be a locally convex Hausdorff space.

- $U \subset X$ **sequentially open**

$$:\Leftrightarrow [(x_n)_{n \in \mathbb{N}} \subset X, x_n \rightarrow x \in U \Rightarrow \exists N \in \mathbb{N} \forall n \geq N : x_n \in U]$$

- X is called **C-sequential space** if each convex sequentially open subset is open.

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Proposition (K, Meichsner, Seifert '21)

Let $(X, \|\cdot\|, \tau)$ be a Saks space. Then:

$$\tau \text{ metrisable on } \|\cdot\| \text{-unit ball} \Rightarrow (X, \gamma) \text{ C-sequential}$$

Theorem (K, Seifert '26)

Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(S(t))_{t \geq 0}$ a semigroup or cosine family on X . Consider the following assertions.

- (a) $(S(t))_{t \geq 0}$ is locally bi-continuous.
- (b) $(S(t))_{t \geq 0}$ is γ -strongly continuous and locally γ -equicontinuous.
- (c) $(S(t))_{t \geq 0}$ is γ -strongly continuous and exponentially γ -equicontinuous.
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Then we have the following.

1. If (X, γ) is C-sequential, then $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.
2. If $\mathcal{N}_\gamma$ is **countably quasi-convex**, then $(b) \Leftrightarrow (c)$. In particular, this holds if $(X, \|\cdot\|)$ is a Banach and (X, γ) a Mackey space.
3. If (X, γ) is a strong Mackey space, then $(b) \Leftrightarrow (c) \Leftrightarrow (d)$.
4. If (X, γ) is a strong Mackey–Mazur space, then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$.

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Here:

$\mathcal{N}_\gamma := \{p: X \rightarrow \mathbb{R} \mid p \text{ } \gamma\text{-cont. seminorm, } p \leq \|\cdot\|\}$ is **countably quasi-convex** if

$$\forall (p_n)_{n \in \mathbb{N}} \subset \mathcal{N}_\gamma, (a_n)_{n \in \mathbb{N}} \subset [0, \infty), \sum_{n=1}^{\infty} a_n = 1 \exists q \in \mathcal{N}_\gamma : \sup_{n \in \mathbb{N}} a_n p_n \leq q.$$

Sketch of the proof

Claim: If $\mathcal{N}_\gamma$ is countably quasi-convex, the following assertions are equivalent.

- (b) $(S(t))_{t \geq 0}$ is γ -strongly continuous and locally γ -equicontinuous.
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Proof:

► $(S(t))_{t \geq 0}$ locally γ -equicontinuous $\Rightarrow$

$$\exists \omega \in \mathbb{R}, M \geq 0 \forall p \in \mathcal{N}_\gamma \exists (q_n)_{n \in \mathbb{N}} \subset \mathcal{N}_\gamma \forall x \in X, n \in \mathbb{N} : \\ \sup_{t \in [0, n]} p(e^{-\omega t} S(t)x) \leq M q_n(x)$$

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- For $\alpha > \omega$ there is a $(a_n)_{n \in \mathbb{N}} \subset [0, \infty)$, $\sum_{n=1}^{\infty} a_n = 1$, such that

$$\sup_{t \geq 0} p(e^{-\alpha t} S(t)x) \leq \frac{M}{1 - e^{\omega - \alpha}} \sup_{n \in \mathbb{N}} a_n q_n(x), \quad x \in X.$$

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- ▶ Use that $\mathcal{N}_\gamma$ is countably quasi-convex.