Semigroups and cosine families on Saks spaces and their relation to bi-continuity

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Motivation

Observation

The semigroups $(S(t))_{t\geqslant 0}$ given by

- $$S(t)f(x) := f(x+t), \quad x\in\mathbb{R},\, f\in \mathrm{C_b}(\mathbb{R}),\, t\geqslant 0,$$
- $$S(t)f(z) := f(\mathrm{e}^{\mathrm{i}t}z), \quad z\in\mathbb{D},\, f\in \mathrm{H}^\infty(\mathbb{D}),\, t\geqslant 0,$$
- $S(0)f := f$ and
$$S(t)f(x) := \tfrac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} f(y)\mathrm{e}^{\frac{-|y-x|^2}{4t}}\,\mathrm{d}y, \quad x\in\mathbb{R}^d,\, f\in \mathrm{C_b}(\mathbb{R}^d),\, t>0,$$

are **not** strongly continuous w.r.t. $\|\cdot\|_\infty$.

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and the cosine family $(S(t))_{t\geqslant 0}$ given by

- $$S(t)f(x) \coloneqq \tfrac{1}{2}\big(f(x+t)+f(x-t)\big), \quad x\in\mathbb{R},\ f\in \mathrm{C_b}(\mathbb{R}),\ t\geqslant 0,$$

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are **not** strongly continuous w.r.t. $\|\cdot\|_\infty$.

However, they are strongly continuous w.r.t. the compact-open topology τ_{co}. They belong to the class of locally bi-continuous semigroups/cosine families.

Saks space & Bi-continuity

Definition (Wiweger 1961, Cooper 1978)

The triple $(X, \|\cdot\|, \tau)$ is called a Saks space if

- $(X, \|\cdot\|)$ is normed, τ a coarser Hausdorff l.c. topology on X, and
- there is a system of seminorms Γ_τ inducing τ such that

$$\|x\| = \sup_{p \in \Gamma_\tau} p(x), \quad x \in X.$$

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Definition (Kühnemund '01, Duan & Sun '06, Budde '24)

Let $(X, \|\cdot\|, \tau)$ be a Saks space. A semigroup or cosine family $(S(t))_{t \geqslant 0}$ in $\mathcal{L}(X)$ is called locally bi-continuous if it is

1. τ-strongly continuous,
2. exponentially bounded, i.e. $\exists\, M \geqslant 1, \omega \in \mathbb{R}\ \forall\, t \geqslant 0 :\ \|S(t)\|_{\mathcal{L}(X)} \leqslant M\mathrm{e}^{\omega t}$,
3. locally bi-equicontinuous, i.e.

$$\forall\, t_0 \geqslant 0,\, p \in \Gamma_\tau, (x_n)_{n \in \mathbb{N}} \subset X,\, x_n \to 0 \text{ w.r.t. } \tau,\, \sup_{n \in \mathbb{N}} \|x_n\| < \infty :$$

$$\sup_{t \in [0, t_0]} p(S(t)x_n) \to 0.$$

Strong continuity and (sequential) equicontinuity

Let

- X be a locally convex Hausdorff space, and
- Γ a system of seminorms inducing its topology.

Definition

A family $(S(t))_{t \geqslant 0}$ of linear maps from X to X is called

- semigroup if $S(0) = \mathrm{id}$ & $S(t)S(s) = S(t+s)$ for all $t, s \geqslant 0$,
- cosine family if $S(0) = \mathrm{id}$ & $2S(t)S(s) = S(t+s) + S(t-s)$ for all $t \geqslant s \geqslant 0$,
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- strongly continuous if $[0, \infty) \ni t \mapsto S(t)x \in X$ is continuous for all $x \in X$,
- locally equicontinuous if

$$\forall\, t_0 \geqslant 0,\ p \in \Gamma\ \exists\, q \in \Gamma,\ C \geqslant 0\ \forall\, x \in X:\ \sup_{t \in [0, t_0]} p(S(t)x) \leqslant C q(x),$$

- exponentially equicontinuous if

$$\exists\, \alpha \in \mathbb{R}\ \forall\, p \in \Gamma\ \exists\, q \in \Gamma,\ C \geqslant 0\ \forall\, x \in X:\ \sup_{t \geqslant 0} p(\mathrm{e}^{-\alpha t} S(t)x) \leqslant C q(x),$$

- sequentially locally equicontinuous if

$$\forall\, t_0 \geqslant 0,\ p \in \Gamma, (x_n)_{n \in \mathbb{N}} \subset X,\ x_n \to 0:\ \sup_{t \in [0, t_0]} p(S(t)x_n) \to 0.$$

Mixed topology

Back to the definition of local bi-equicontinuity:

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Observation (Wiweger 1961):

- There is a l.c. Hausdorff topology $\gamma := \gamma(\|\cdot\|, \tau)$ such that:

$$\gamma\text{-}\lim_{n\to\infty} x_n = x \quad \Leftrightarrow \quad \tau\text{-}\lim_{n\to\infty} x_n = x \text{ and } \sup_{n\in\mathbb{N}} \|x_n\| < \infty$$

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Example (Buck 1958, Cooper 1971)

- Let $X := \mathrm{C_b}(\mathbb{R}^d)$ with $\|\cdot\| := \|\cdot\|_\infty$ and $\tau := \tau_{\mathrm{co}}$.
- Then γ is generated by the seminorms

$$|f|_\nu := \sup_{x\in\mathbb{R}^d} |f(x)\nu(x)|, \quad f \in X,\ \nu \in \mathrm{C_0}(\mathbb{R}^d).$$

Bi-continuity & Mixed topology

Corollary

Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(S(t))_{t\geqslant 0}$ a semigroup or cosine family on X. Then the following assertions are equivalent.

1. $(S(t))_{t\geqslant 0}$ is locally bi-continuous.
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Theorem (Kraaij '16)

Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(S(t))_{t\geqslant 0}$ a semigroup on X.
If (X, γ) is C-sequential, then the following assertions are equivalent.

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Definition (Snipes 1973)

Let X be a locally convex Hausdorff space.

- $U \subset X$ sequentially open

$$:\Leftrightarrow \ [\,(x_n)_{n\in\mathbb{N}} \subset X,\, x_n \to x \in U \ \Rightarrow \ \exists\, N \in \mathbb{N}\, \forall\, n \geqslant N :\ x_n \in U\,]$$

- X is called C-sequential space if each convex sequentially open subset is open.

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Proposition (K, Meichsner, Seifert '21)

Let $(X, \|\cdot\|, \tau)$ be a Saks space. Then:

$$\tau \text{ metrisable on } \|\cdot\|\text{-unit ball} \;\Rightarrow\; (X, \gamma) \text{ C-sequential}$$

Theorem (K, Seifert '26)

Let $(X, \|\cdot\|, \tau)$ be a Saks space and $(S(t))_{t\geqslant 0}$ a semigroup or cosine family on X. Consider the following assertions.

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Then we have the following.

1. If (X, γ) is C-sequential, then (a)⇔(b)⇔(c).
2. If $\mathcal{N}_\gamma$ is countably quasi-convex, then (b)⇔(c). In particular, this holds if $(X, \|\cdot\|)$ is a Banach and (X, γ) a Mackey space.
3. If (X, γ) is a strong Mackey space, then (b)⇔(c)⇔(d).
4. If (X, γ) is a strong Mackey–Mazur space, then (a)⇔(b)⇔(c)⇔(d).

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Here:
$\mathcal{N}_\gamma \coloneqq \{p \colon X \to \mathbb{R} \mid p\ \gamma\text{-cont. seminorm}, p \leqslant \|\cdot\|\}$ is countably quasi-convex if

$$\forall\ (p_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\gamma,\ (a_n)_{n\in\mathbb{N}} \subset [0,\infty),\ \sum_{n=1}^{\infty} a_n = 1\ \exists\ q \in \mathcal{N}_\gamma :\ \sup_{n\in\mathbb{N}} a_n p_n \leqslant q\,.$$

Sketch of the proof

Claim: If $\mathcal{N}_\gamma$ is countably quasi-convex, the following assertions are equivalent.

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Proof:

- $(S(t))_{t\geqslant 0}$ locally γ-equicontinuous $\Rightarrow$

$$\exists\, \omega \in \mathbb{R}, M \geqslant 0\; \forall\, p \in \mathcal{N}_\gamma\; \exists\, (q_n)_{n\in\mathbb{N}} \subset \mathcal{N}_\gamma\; \forall\, x \in X,\, n \in \mathbb{N} :$$
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- Use that $\mathcal{N}_\gamma$ is countably quasi-convex.