Spectral theory for semigroups on locally convex spaces

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Motivation (Engel & Nagel 2000)

- Let X be a Banach space, $A\colon D(A) \subseteq X \to X$ a closed linear operator, and
- consider the abstract Cauchy problem

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  1. the spectral bound $s(A) := \sup\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\} < 0$, and
  2. the spectral mapping theorem holds, i.e.

$$\sigma(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma(A)}, \quad t \geqslant 0,$$

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- Question: What if X is **not** a Banach space?

Spectra

Let X be a locally convex Hausdorff space (lcHs) and $A\colon D(A) \subseteq X \to X$ a closed linear operator. Then the spectrum of A is defined by

$$\sigma(A) \coloneqq \mathbb{C} \setminus \rho(A) \coloneqq \mathbb{C} \setminus \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is bijective and } (\lambda - A)^{-1} \in \mathcal{L}(X)\},$$

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the point spectrum by

$$\sigma_{\mathrm{p}}(A) \coloneqq \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is not injective}\},$$

the approximate point spectrum by

$$\sigma_{\mathrm{ap}}(A) \coloneqq \{\lambda \in \mathbb{C} \mid \exists \text{ a net } (x_i)_{i \in I} \subseteq D(A) \text{ not conv. to } 0 : \lim_{i \in I} (A - \lambda) x_i = 0\},$$

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$$\sigma_{\mathrm{r}}(A) := \{\lambda \in \mathbb{C} \mid \operatorname{ran}(\lambda - A) \text{ is not dense in } X\},$$

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Spectral decomposition (K 2026)

1. If X is complete, then $\sigma(A) = \sigma_{\mathrm{ap}}(A) \cup \sigma_{\mathrm{r}}(A) \cup \sigma_{\mathrm{t}}(A)$.
2. If X is Fréchet, then $\sigma_{\mathrm{ap}}^{\mathrm{seq}}(A) = \sigma_{\mathrm{ap}}(A)$.

Strong continuity, local equicontinuity and generators

Let X be an lcHs and Γ a system of seminorms inducing its topology.

Definition

A family $(T(t))_{t\geqslant 0}$ in $\mathcal{L}(X)$ is called

- semigroup if $T(0) = \mathrm{id}$ & $T(t)T(s) = T(t+s)$ for all $t, s \geqslant 0$,
- strongly continuous if $[0,\infty) \ni t \mapsto T(t)x \in X$ is continuous for all $x \in X$,
- locally equicontinuous if

$$\forall\, t_0 \geqslant 0,\ p \in \Gamma\ \exists\, q \in \Gamma,\, C \geqslant 0\ \forall\, x \in X : \sup_{t\in[0,t_0]} p(T(t)x) \leqslant Cq(x).$$

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- If $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup, its generator A is

$$Ax \coloneqq \lim_{t\to 0+} \tfrac{T(t)x-x}{t}, \quad x \in D(A) \coloneqq \Big\{ y \in X \mid \lim_{t\to 0+} \tfrac{T(t)y-y}{t} \text{ exists in } X \Big\}.$$

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Proposition (Kōmura 1968)

Let $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X.

1. If X is sequentially complete, then $D(A)$ is dense in X.
2. If $(T(t))_{t\geqslant 0}$ is locally equicontinuous, then A is closed.

Spectral inclusions

Spectral inclusion theorem (K 2026)

If

- X is a sequentially complete lcHs,
- $(T(t))_{t\geqslant 0}$ a strongly continuous locally equicontinuous semigroup on X with generator A,

then

$$\mathrm{e}^{t\sigma_*(A)} \subseteq \sigma_*(T(t)), \quad t \geqslant 0,$$

for all the spectra σ_* introduced before (and some more, apart from $* = \mathrm{t}$).

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Sketch of proof:
Use for all $\lambda \in \mathbb{C}$ and $t \geqslant 0$ that

$$\begin{aligned} \mathrm{e}^{-\lambda t}T(t)x - x &= (A-\lambda)\int_0^t \mathrm{e}^{-\lambda s}T(s)x\,\mathrm{d}s && \text{if } x \in X, \\ &= \int_0^t \mathrm{e}^{-\lambda s}T(s)(A-\lambda)x\,\mathrm{d}s && \text{if } x \in D(A). \end{aligned}$$

Point spectrum

Spectral mapping theorem for the point spectrum (K 2026)

If

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- Only need to show $\supseteq$ by spectral inclusion theorem, and assume $X \neq \{0\}$.
- Let $\lambda \in \sigma_{\mathrm{p}}(T(t))$, $\lambda \neq 0$, and $t \geqslant 0$. Reduce it with a rescaling to the case $\lambda = 1$ and $t = 1$.

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- Consider restricted semigroup $(T(s)_{|Y})_{s\geqslant 0}$ on the eigenspace

$$Y := \{y \in X \mid T(1)y = y\}.$$

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- Use that $(T(s)_{|Y})_{s\geqslant 0}$ is periodic with minimal period $\rho \leqslant 1$, and that $\varnothing \neq \sigma_{\mathrm{p}}(A_{|Y}) \subseteq 2\pi\mathrm{i}k\mathbb{Z}$ for some $k \in \mathbb{N}$ if $\rho > 0$.

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- If $\rho = 0$, then $A_{|Y} = 0$ and $0 \in \sigma_{\mathrm{p}}(A_{|Y})$.

Residual spectrum

Recall:

- If X is complete, then $\sigma(A) = \sigma_{\mathrm{ap}}(A) \cup \sigma_{\mathrm{r}}(A) \cup \sigma_{\mathrm{t}}(A)$.
- If X is sequentially complete, then $D(A)$ is dense in X.

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Spectral mapping theorem for the residual spectrum (K 2026)

If

- X is a sequentially complete lcHs such that X'_{b} is sequentially complete,
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- Kōmura 1968: $(T^{\odot}(t))_{t\geqslant 0}$ is str. cont. loc. equicont. on $(X^{\odot}, \beta(X',X))$.
- Apply the spectral mapping theorem for the point spectrum to $(T^{\odot}(t))_{t\geqslant 0}$.

Approximate point spectrum

Recall: The approximate point spectrum is defined by
$\sigma_{\mathrm{ap}}(A) \coloneqq \{\lambda \in \mathbb{C} \,|\, \exists$ a net $(x_i)_{i\in I} \subseteq D(A)$, $x_i \not\to 0 : \lim_{i\in I}(A-\lambda)x_i = 0\}$.

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Spectral mapping theorem for the bounded approximate point spectrum (K 2026)

If

- X is a semi-Montel space,
- $(T(t))_{t\geqslant 0}$ a strongly continuous locally equicontinuous semigroup on X with generator A,

then

$$\sigma_{\mathrm{bap}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{bap}}(A)} \quad \text{and} \quad \sigma_{\mathrm{bap}}^{\mathrm{seq}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{bap}}^{\mathrm{seq}}(A)}, \quad t \geqslant 0.$$

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Remark:

- Not known whether one can drop the boundedness assumption.

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- $(T(t))_{t\geqslant 0}$ a strongly continuous locally equicontinuous semigroup on X with generator A,

then

$$\sigma_{\mathrm{bap}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma_{\mathrm{bap}}(A)} \quad \text{and} \quad \sigma^{\mathrm{seq}}_{\mathrm{bap}}(T(t)) \setminus \{0\} = \mathrm{e}^{t\sigma^{\mathrm{seq}}_{\mathrm{bap}}(A)}, \quad t \geqslant 0.$$

Remark:

- Not known whether one can drop the boundedness assumption.
- The sequential version also holds if one assumes instead of X being semi-Montel that $(T(t))_{t\geqslant 0}$ is eventually uniformly continuous.